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On the Limits of (Linear Combinations of) Iterates of Linear Operators

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We give necessary and sufficient conditions such that iterates or certain linear combinations of iterates of linear operators of finite dimensional range, respectively, converge. In case of convergence, we give an expression for the limit as well as estimates for the rate of convergence. Our results are then applied to Schoenberg type and Sablonnière operators as well as tensor product Schoenberg type operators, Bernstein and Bernstein–Durrmeyer operators over triangles.

1. INTRODUCTION AND MAIN RESULTS

Iterates of univariate Bernstein operators were studied by Kelisky and Rivlin in 1967 [13]. One of the results of their paper was the following:

$$\lim_{M \to \infty} B_n^M(f; x) = f(0) + (f(1) - f(0))\, x, \qquad 0 \leqslant x \leqslant 1, \tag{1}$$

where f is a continuous function on $[0, 1]$, $B_n(f; \cdot)$ is the Bernstein polynomial of degree n associated with f, and where the iterates of the Bernstein polynomial are defined by

$$B_n^M(f; \cdot) := B_n(B_n^{M-1}(f; \cdot); \cdot), \qquad M = 2, 3, ...$$

Thus, (1) says that the iterates of the Bernstein polynomial converge to the linear function that interpolates f at the points 0 and 1. Generalizations of the results of Kelisky and Rivlin to different directions were made by several authors (see Chen and Feng [5] for an overview).

On the other hand side, in 1973 Micchelli [15] introduced certain linear combinations of iterates of univariate Bernstein operators. These linear combinations can also be regarded as iterated Boolean sums $\bigoplus^M B_n(f; \cdot)$,

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$M \geqslant 0$, as is done, e.g., in [9]. We will adopt this notation for shortness in the present paper. The Boolean sum of two operators A and B is given by

$$A \oplus B := A + B - A \circ B,$$

and the iterated Boolean sum of an operator B is defined to be

$$\bigoplus^0 B := id, \qquad \bigoplus^1 B := B,$$
$$\bigoplus^{M+1} B := B \oplus \left(\bigoplus^M B\right), \qquad M \geqslant 1.$$

After 1973, iterated Boolean sums of univariate Bernstein operators were studied by other authors (see [9] for an overview). In 1995 Sevy [17] found that the limit of iterated Boolean sums of Bernstein polynomials is the interpolation polynomial with respect to the nodes $(i/n, f(i/n))$, $i = 0, ..., n$, and that the limit of iterated Boolean sums of Bernstein–Durrmeyer operators is the least squares polynomial with respect to the $L_2[0, 1]$-norm.

The results of Kelisky/Rivlin and Sevy can be generalized once we use some results from linear algebra. In order to do so, we introduce the following notation: In a normed linear space X, a linear operator $B: X \to X$ with $(n+1)$-dimensional range $B(X) = \text{span}(b_0, ..., b_n)$ is represented by

$$Bf := \sum_{i=0}^{n} \lambda_i f b_i, \tag{2}$$

where $\lambda_0, ..., \lambda_n$ are linear functionals on X. With this, we denote the Gramian associated with B by

$$\mathscr{B} := (\lambda_i b_j)_{i,j=0}^{n}. \tag{3}$$

As a generalization of the result of Kelisky/Rivlin above we state:

THEOREM 1. *Let X be a normed linear space and B: $X \to X$ be a linear operator with $(n+1)$-dimensional range $B(X) = \text{span}(b_0, ..., b_n)$. With the operator B as in* (2) *and the Gramian $\mathscr{B}$ associated with B as in* (3), *the following characterization holds*:

The iterates $B^M f := B(B^{M-1} f)$ converge for all $f \in X$ if and only if the sequence of powers of the Gramian $\mathscr{B}$ converges (*i.e.* $\lim_{M \to \infty} \mathscr{B}^M$ *exists*).

In this case, $\mathbf{f} := (\lambda_i f)_{i=0}^{n}$ has the unique representation $\mathbf{f} = \mathbf{v}_1 + \mathbf{v}_0$ where $\mathbf{v}_1$ is an eigenvector with respect to eigenvalue 1 *and $\mathscr{B}$ and $\mathbf{v}_0$ is a linear combination of generalized eigenvectors with respect to eigenvalues of modulus less than* 1. *Then we have*

$$Lf := \lim_{M \to \infty} B^M f = \sum_{i=0}^{n} (\mathbf{v}_1)_i \, b_i. \tag{4}$$

Remarks. (1) Matrices $\mathscr{B}$ for which $\lim_{M\to\infty}\mathscr{B}^M$ exists are called *semiconvergent* (see Definition 2 below) and are treated, e.g., in the book of Berman and Plemmons [1].

(2) If 1 is not an eigenvalue of $\mathscr{B}$ or if $\mathbf{v}_1=0$, (4) is understood as $Lf=0$.

(3) An easy exercise shows (see [1], p. 198) that in case that the eigenvectors of $\gamma:=\max\{|l| : l \text{ is eigenvalue of } \mathscr{B},\ l\neq 1\}$ are all proper eigenvectors we have

$$\|Lf-B^Mf\|\leqslant C\cdot\gamma^M \tag{5}$$

with a constant C which does not depend on M and $\|\cdot\|$ any norm on $\operatorname{span}\{b_0, ..., b_n\}$. In other cases, (5) is a little different. However, for the examples considered below, this estimate holds.

The subsequent theorem is a generalization of Sevy's result:

THEOREM 2. *Let X be a normed linear space and $B : X\to X$ as in* (2) *be a linear operator with* $(n+1)$*-dimensional range* $B(X)=\operatorname{span}(b_0, ..., b_n)$. *With $\varrho(A)$ being the spectral radius of a matrix A and I denoting the identity matrix, we have the following: If $\varrho(I-\mathscr{B})<1$ is true, then the iterated Boolean sums $\bigoplus^M Bf$ converge for all $f\in X$ and $\mathscr{B}$ is nonsingular. In this case, $L^{\oplus}f:=\lim_{M\to\infty}\bigoplus^M Bf$ is the unique interpolant in $B(X)$ that satisfies the interpolation problem*

$$\lambda_i(f-L^{\oplus}f)=0, \qquad i=0, ..., n, \tag{6}$$

and has the representation

$$L^{\oplus}f=\sum_{i=0}^{n}(\mathscr{B}^{-1}\mathbf{f})_i\, b_i. \tag{7}$$

Conversely, if $\bigoplus^M Bf$, $M\to\infty$, converges for all $f\in X$, we have $\varrho(I-\mathscr{B})<1$.

Remarks. (1) From the proof of Theorem 2 it will become clear that Boolean sums of linear operators B have a close relation to partial sums of the formal Neumann series for $\mathscr{B}^{-1}=(I-(I-\mathscr{B}))^{-1}$.

(2) Again, it is an easy exercise to show that in case of convergence, for $\operatorname{rank}(I-\mathscr{B})=\operatorname{rank}(I-\mathscr{B})^2$ the estimate

$$\|L^{\oplus}f-\textstyle\bigoplus^M Bf\|\leqslant C\cdot\gamma^M \tag{8}$$

holds with a constant C which does not depend on M, $\gamma:=\varrho(I-\mathscr{B})$ and $\|\cdot\|$ any norm on $\operatorname{span}\{b_0, ..., b_n\}$. As above, the estimate is a little different for $\operatorname{rank}(I-\mathscr{B})\neq\operatorname{rank}(I-\mathscr{B})^2$, but it is true for the applications below.

The paper is organized as follows: In Section 2, we apply Theorems 1 and 2 to the univariate spline operators of Schoenberg and Sablonnière. In Section 3, application to the tensor product Schoenberg type operator, the Bernstein operator and the Bernstein–Durrmeyer operator over triangles is provided. Finally we give the proofs to the results of Sections 1–3 in Section 4.

2. APPLICATIONS IN ONE VARIABLE

We define kth order B-splines $N_{i,k,t}$, $i=-k+1, ..., n-1$, over the knot sequence $t=(t_i)_{i=-k+1}^{n+k-1} \in \mathbb{R}^{n+2k-1}$ with

$$\begin{aligned} &t_{-k+1} = \cdots = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = \cdots = t_{n+k-1}, \\ &t_i < t_{i+k-1}, \qquad i=-k+2, ..., n-1, \end{aligned} \tag{9}$$

by the following relation:

$$N_{i,k,t}(x) := (t_{i+k} - t_i)[t_i, ..., t_{i+k}](\cdot - x)_+^{k-1}, \qquad x \in [t_0, t_n]. \tag{10}$$

Here, $[t_i, ..., t_{i+k}]\, f(\cdot)$ denotes the divided difference of a function f with respect to the knots $t_i, ..., t_{i+k}$, and f_+ is defined pointwise by $f_+(x) := \max\{f(x), 0\}$, $x \in \mathbb{R}$. We assume $N_{i,k,t}$, $i=-k+1, ..., n-1$, to be left continuous and $-$ in order to have a partition of unity on $[t_0, t_n] - N_{n-1,k,t}$ to be also right continuous.

The conditions (9) are slightly different from the usual ones for spline knots where $t_i < t_{i+k}$, $i=-k+1, ..., n-1$, is required. However, our condition is not really stronger, since for $t_i = t_{i+k-1}$ the intervals $[t_0, t_i]$ with knot sequence $t_{-k+1}, ..., t_{i+k-1}$ and $[t_{i+k-1}, t_n]$ with knot sequence $t_i, ..., t_{n+k-1}$ can be considered separately.

Using B-splines we define the operators B_I for continuous functions on $[t_0, t_n]$:

$$B_I f := \sum_{i=-k+1}^{n-1} f(\tau_i) N_{i,k,t}. \tag{11}$$

Here, the nodes τ_i are required to be strictly increasing and to satisfy

$$\left.\begin{aligned} &t_{i+1} < \tau_i < t_{i+k-1}, && \text{if} \quad t_{i+1} < t_{i+k-1}, \\ &\tau_i = t_{i+k-1}, && \text{if} \quad t_{i+1} = t_{i+k-1}, \end{aligned}\right\} \qquad i=-k+1, ..., n-1. \tag{12}$$

For $\tau_i = t_i^* := (t_{i+1} + \cdots + t_{i+k-1})/(k-1)$, $i=-k+1, ..., n-1$, the operator B_I of (11) is Schoenberg's variation diminishing spline (see [3], p. 159 ff.) which is frequently used for curve design in CAD systems.

As a first application of the theorems of Section 1 we can show the subsequent result:

THEOREM 3. *For the operators* B_I *given in* (11)–(12) *with order* $k>1$ *and a function* $f \in C[t_0, t_n]$, *the limits* $L_I f := \lim_{M\to\infty} B_I^M f$ *and* $L_I^{\oplus} f := \lim_{M\to\infty} \bigoplus^M B_I f$ *exist. Moreover, the following is true*:

(i) *If* $B_I f$ *is Schoenberg's variation diminishing spline, then*

$$L_I f(x) = f(t_0) + \frac{x-t_0}{t_n-t_0}\,(f(t_n)-f(t_0)), \qquad x \in [t_0, t_n].$$

(ii) $L_I^{\oplus} f$ *is the unique spline of order* k *with respect to the knot sequence* t *that interpolates* f *at the nodes* τ_i, $i=-k+1, ..., n-1$.

Remark. For the knot sequence t with $t_{-k+1} = \cdots = t_0 = 0$, $t_1 = \cdots = t_k = 1$, the B-splines are Bernstein basis polynomials, and we have $t_i^* = (i+k-1)/(k-1)$, $i=-k+1, ..., 0$. Thus, Schoenberg's variation diminishing spline and the Bernstein polynomial are the same in this case, and Theorem 3 includes the result of Kelisky/Rivlin and Sevy's result concerning the Bernstein polynomial. Moreover, from Kelisky/Rivlin [13], (1.6), and from equations (5) and (8) we get the following estimates for Bernstein polynomials:

$$\|L_I f - B_I^M f\| \leqslant C \cdot \left(\frac{k-2}{k-1}\right)^M$$

$$\|L_I^{\oplus} f - \bigoplus^M B_I f\| \leqslant \tilde{C} \cdot \left(1 - \frac{(k-2)!}{(k-1)^{k-2}}\right)^M.$$

Here, $\|\cdot\|$ is any norm on $\pi_k[0, 1]$.

Sablonnière [16] introduced another family of spline operators, which apply to functions $f \in L_p[t_0, t_n]$, $1 \leqslant p \leqslant \infty$:

$$B_{II} f := \sum_{i=-k+1}^{n-1} \left(\frac{k}{t_{i+k}-t_i} \int_{t_0}^{t_n} N_{i,k,t}(x)\, f(x)\, dx\right) N_{i,k,t}. \tag{13}$$

These operators generalize the modified Bernstein operators of Durrmeyer [6].

Here we present a result for Sablonnière's operators that is similar to Theorem 3:

THEOREM 4. *For operators B_{II} as in* (13) *and a function $f \in L_p[t_0, t_n]$, $1 \leqslant p \leqslant \infty$, we have*:

(i) $$\lim_{M \to \infty} B_{II}^M f \equiv \text{const} = \frac{k}{n+k-1} \sum_{i=-k+1}^{n-1} \frac{1}{t_{i+k} - t_i} \int_{t_0}^{t_n} N_{i,k,t}(x)\, f(x)\, dx.$$

(ii) $\lim_{M \to \infty} \bigoplus^M B_{II} f$ *is the unique least squares spline approximant to f from* $\operatorname{span}\{N_{i,k,t} : i = -k+1, ..., n-1\}$ (*with respect to the $L_2[t_0, t_n]$-norm*).

Remark. For the special choice $t_{-k+1} = \cdots = t_0 = 0$ and $t_1 = \cdots = t_k = 1$ for the spline knots, Sablonnière's operators coincide with Durrmeyer's operators. Therefore, the subsequent corollary holds, part (ii) of which can be found in the paper of Sevy [17]:

COROLLARY 5. *With the Bernstein–Durrmeyer operators D_k, defined for $f \in L_p[0, 1]$, $1 \leqslant p \leqslant \infty$, by*

$$D_k f := k \sum_{i=0}^{k-1} \int_0^1 b_{i,k}(x)\, f(x)\, dx\, b_{i,k}$$

$$b_{i,k}(x) := \binom{k-1}{i} x^i (1-x)^{k-1-i}, \qquad i = 0, ..., k-1,$$

we have

(i) $L_{II} f(x) := \lim_{M \to \infty} D_k^M f(x) \equiv \text{const} \equiv \int_0^1 f(x)\, dx$.

(ii) $L_{II}^{\oplus} f := \lim_{M \to \infty} \bigoplus^M D_k f$ *is the unique least squares approximant to f from π_k, the space of polynomials of order not exceeding k* (*here, least squares means with respect to the $L_2[0, 1]$-norm*).

(iii) *with any norm $\| \cdot \|$ on $\pi_k[0, 1]$ there holds*:

$$\| L_{II} f - D_k^M f \| \leqslant C \cdot \left(\frac{k-1}{k+1} \right)^M$$

$$\| L_{II}^{\oplus} f - \bigoplus^M D_k f \| \leqslant \tilde{C} \cdot \left(1 - \binom{2k-1}{k}^{-1} \right)^M.$$

3. APPLICATIONS IN MORE THAN ONE VARIABLE

In this section, we have a look at three families of operators in two variables. It should become clear that the general setting in more than two variables is to be treated along the same lines, but is simply a little more technical.

The first family of operators to be considered here is the tensor product spline operators of Schoenberg type: Given the knot sequences $t=(t_i)_{i=-k+1}^{n+k-1}$ and $\tilde{t}=(\tilde{t}_i)_{i=-\tilde{k}+1}^{\tilde{n}+\tilde{k}-1}$, we define the B-splines $N_{i,k,t}$ and $N_{i,\tilde{k},\tilde{t}}$ of order k and $\tilde{k}$, respectively, as in Section 2. For the node sequences $\tau=(\tau_i)_{i=-k+1}^{n-1}$ and $\tilde{\tau}=(\tilde{\tau}_i)_{i=-\tilde{k}+1}^{\tilde{n}-1}$, we require (12) in either case. Then we can define the operator B_T by

$$\begin{aligned} &B_T : C([t_0, t_n]\times[\tilde{t}_0, \tilde{t}_{\tilde{n}}]) \to C([t_0, t_n]\times[\tilde{t}_0, \tilde{t}_{\tilde{n}}]) \\ &B_T f := \sum_{i=-k+1}^{n-1} \sum_{j=-\tilde{k}+1}^{\tilde{n}-1} f(\tau_i, \tilde{\tau}_j)\cdot N_{i,k,t} N_{j,\tilde{k},\tilde{t}}. \end{aligned} \tag{14}$$

The theorem below shows that the limits of B_T's iterates and B_T's iterated Boolean sums look as we expected:

THEOREM 6. *For the tensor product Schoenberg operator B_T of* (14), *we have*

(i) $$\lim_{M\to\infty} B_T^M f(x, y) = \frac{\tilde{t}_{\tilde{n}}-y}{\tilde{t}_{\tilde{n}}-\tilde{t}_0}\left(\frac{t_n-x}{t_n-t_0} f(t_0, \tilde{t}_0)+\frac{x-t_0}{t_n-t_0} f(t_n, \tilde{t}_0)\right) + \frac{y-\tilde{t}_0}{\tilde{t}_{\tilde{n}}-\tilde{t}_0}\left(\frac{t_n-x}{t_n-t_0} f(t_0, \tilde{t}_{\tilde{n}})+\frac{x-t_0}{t_n-t_0} f(t_n, \tilde{t}_{\tilde{n}})\right)$$

(ii) $\lim_{M\to\infty} \bigoplus^M B_T f$ *is the unique interpolant to f in*

$$\operatorname{span}\{N_{i,k,t}\cdot N_{j,\tilde{k},\tilde{t}} : i=-k+1, ..., n-1,\ j=-\tilde{k}+1, ..., \tilde{n}-1\}$$

with respect to the nodes $(\tau_i, \tilde{\tau}_j)$, $i=-k+1, ..., n-1$, $j=-\tilde{k}+1, ..., \tilde{n}-1$.

Remark. An analoguous result for tensor products of Sablonnière's operators can be proved by the same method as is used in the proof of Theorem 6.

Our second bivariate example deals with the Bernstein operator over triangles which was introduced by Lorentz [14]: Let $S := \{(x, y) : x, y \geqslant 0, x+y \leqslant 1\}$ be the standard triangle in $\mathbb{R}^2$. Then for $f\in C(S)$, the Bernstein operator of total order k over S is defined by

$$B_{I,\Delta} f := \sum_{0\leqslant i+j<k} f(i/(k-1), j/(k-1))\, b_{ij}, \tag{15}$$

where the Bernstein basis functions b_{ij}, $0\leqslant i+j<k$, are given by

$$b_{ij}(x, y) := \frac{(k-1)!}{i!\, j!\, (k-1-i-j)!}\, x^i y^j (1-x-y)^{k-1-i-j}, \quad (x, y)\in S. \tag{16}$$

The result that deals with this operator also profits from Theorems 1 and 2 of Section 1. Part (i) of it was proved already by Chang and Feng in 1986 [4] in a way different from ours.

THEOREM 7. *For the Bernstein operator over the standard triangle, $B_{I,\Delta}$, according to* (15) *and* (16) *and for $f \in C(S)$, we have*

(i) $L_{I,\Delta} f(x, y) := \lim_{M \to \infty} B^M_{I,\Delta} f(x, y) = f(1, 0)\, x + f(0, 1)\, y + (1 - x - y)\, f(0, 0)$.

(ii) $L^{\oplus}_{I,\Delta} f := \lim_{M \to \infty} \bigoplus^M B_{I,\Delta} f$ *is the unique bivariate interpolation polynomial of total order k with respect to the data*

$$\{(i/(k-1),\ j/(k-1),\ f(i/(k-1),\ j/(k-1))) : 0 \leqslant i + j < k\}.$$

(iii) *with any norm* $\|\cdot\|$ *on* $\pi^2_k(S)$, *there holds*

$$\|L_{I,\Delta} f - B^M_{I,\Delta} f\| \leqslant C \cdot \left(\frac{k-2}{k-1}\right)^M$$

$$\|L^{\oplus}_{I,\Delta} f - \bigoplus^M B_{I,\Delta} f\| \leqslant \tilde{C} \cdot \left(1 - \frac{(k-1)!}{(k-1)^{k-1}}\right)^M.$$

For $f \in L_p(S)$, $1 \leqslant p \leqslant \infty$, Derriennic [8] introduced Durrmeyer operators over simplices which read as follows in two dimensions:

$$B_{II,\Delta} f := k(k+1) \sum_{0 \leqslant i + j < k} \left(\int_S b_{ij}(x, y)\, f(x, y)\, dx\, dy \right) b_{ij}. \qquad (17)$$

Here, b_{ij} are the Bernstein basis functions as in (16). For these operators, we can state and prove our last result:

THEOREM 8. *For the Durrmeyer operator over the standard triangle, $B_{II,\Delta}$, according to* (17) *and* (16) *and for $f \in L_p(S)$, $1 \leqslant p \leqslant \infty$, we have*

(i) $L_{II,\Delta} f(x, y) := \lim_{M \to \infty} B^M_{II,\Delta} f(x, y) = \int_S f(x, y)\, dx\, dy$.

(ii) $L^{\oplus}_{DII} f := \lim_{M \to \infty} \bigoplus^M B_{II,\Delta} f$ *is the unique least squares approximation polynomial to f from* $\pi^2_k(S)$, *the space of bivariate polynomials of total order k* (*here, least squares means with respect to the* $L_2(S)$ *norm*).

(iii) *with any norm* $\|\cdot\|$ *on* $\pi^2_k(S)$, *there holds*

$$\|L_{II,\Delta} f - B^M_{II,\Delta} f\| \leqslant C \cdot \left(\frac{k-1}{k+2}\right)^M$$

$$\|L^{\oplus}_{II,\Delta} f - \bigoplus^M B_{II,\Delta} f\| \leqslant \tilde{C} \cdot \left(1 - \binom{2k}{k+1}^{-1}\right)^M.$$

4. THE PROOFS

Before we prove our results, we introduce some notation and list some definitions and facts from matrix theory which will be useful in the sequel.

For finite linear combinations $\sum_{i=0}^{n} a_i b_i$ where $a_i \in \mathbb{R}$ and b_i are functions, we often write

$$\sum_{i=0}^{n} a_i b_i =: \mathbf{b}^T \mathbf{a} \qquad \text{with} \quad \mathbf{a} := (a_i)_{i=0}^{n} \text{ and } \mathbf{b} := (b_i)_{i=0}^{n}.$$

PROPOSITION 1 [18], Th. 3.7. *If A is an arbitrary complex $(n+1) \times (n+1)$ matrix with $\varrho(A) < 1$, then $I - A$ is nonsingular, and*

$$(I-A)^{-1} = I + A + A^2 + \cdots,$$

the series on the right converging. Conversely, if the series on the right converges, then $\varrho(A) < 1$.

DEFINITION 2 [1], Def. 6.4.8. A matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is called *semi-convergent* whenever $\lim_{M \to \infty} A^M$ exists.

PROPOSITION 3 [1], Ex. 6.4.9. *Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Then A is semi-convergent if and only if each of the following conditions hold*:

(1) $\varrho(A) \leqslant 1$ *($\varrho(A)$ is the spectral radius of A)*,

(2) $\operatorname{rank}(I-A)^2 = \operatorname{rank}(I-A)$,

(3) *if $\varrho(A) = 1$, then every eigenvalue of A with modulus* 1 *equals* 1.

PROPOSITION 4 [1], Lemma 7.6.9. *Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Then A is semiconvergent if and only if the Jordan normal form of A is*

$$A = V \begin{pmatrix} I & 0 \\ 0 & \mathscr{K} \end{pmatrix} V^{-1}, \tag{18}$$

where the identity matrix I is missing if 1 *is not an eigenvalue of A and where $\varrho(\mathscr{K}) < 1$.*

DEFINITION 5 [1], Ex. 2.6.25. A matrix A is called *totally nonnegative* if all its minors of any order are nonnegative.

PROPOSITION 6 (cp. [1], Ex. 2.6.25 and Ex. 2.6.28). *A nonsingular, totally nonnegative matrix $A = (a_{ij})_{i,j=0}^{n} \in \mathbb{R}^{(n+1)\times(n+1)}$ with $a_{ij} > 0$ for $|i-j| \leqslant 1$ has $n+1$ distinct positive eigenvalues.*

Proposition 7 [18], Lemma 2.5. *Let* $A = (a_{ij})_{i,j=0}^{n} \in \mathbb{R}^{(n+1)\times(n+1)}$ *be irreducible and let all its entries* a_{ij} *be nonnegative. Then with* $z_i := \sum_{j=0}^{n} a_{ij}$ *there is either* $z_i = \varrho(A)$, $i = 0, ..., n$, *or* $\min_{0 \leqslant i \leqslant n} z_i < \varrho(A) < \max_{0 \leqslant i \leqslant n} z_i$.

Proof of Theorem 1. We can write

$$Bf = \sum_{i=0}^{n} \lambda_i f b_i = \mathbf{b}^T \mathbf{f}, \tag{19}$$

with $\mathbf{f} = (\lambda_i f)_{i=0}^{n}$ and $\mathbf{b} = (b_i)_{i=0}^{n}$ the vector of basis functions in $B(X)$. Thus, by the linearity of the functionals λ_i, $i = 0, ..., n$,

$$B^2 f = \sum_{i=0}^{n} \left(\lambda_i \left(\sum_{j=0}^{n} \lambda_j f b_j \right) \right) b_i = \sum_{i=0}^{n} b_i \sum_{j=0}^{n} (\lambda_i b_j)(\lambda_j f) = \mathbf{b}^T \mathscr{B} \mathbf{f},$$

with $\mathscr{B}$ as in (3), and induction shows

$$B^M f = \mathbf{b}^T \mathscr{B}^{M-1} \mathbf{f}, \qquad M \geqslant 1. \tag{20}$$

This implies that $B^M f$ converges for all $f \in X$ if and only if $\mathscr{B}^{M-1}$ converges.

From Proposition 4, we see that in this case $\mathscr{B}$ can be written as in (18), where the columns of V are generalized eigenvectors of $\mathscr{B}$. Now (18) allows to write $\mathbf{f} = \mathbf{v}_1 + \mathbf{v}_0$ as stated in the theorem we are to prove. Therefore, with $\varrho(\mathscr{K}) < 1$, we have

$$\begin{aligned} \lim_{M \to \infty} B^M f &= \mathbf{b}^T V \begin{pmatrix} I & 0 \\ 0 & \lim_{M \to \infty} \mathscr{K}^M \end{pmatrix} V^{-1} (\mathbf{v}_1 + \mathbf{v}_0) \\ &= \mathbf{b}^T V \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V^{-1} (\mathbf{v}_1 + \mathbf{v}_0) \\ &= \mathbf{b}^T \mathbf{v}_1, \end{aligned}$$

where I is missing if 1 is not an eigenvalue of $\mathscr{B}$. This equals (4). ▮

Proof of Theorem 2. We can show the following representation for iterated Boolean sums of finite dimensional operators by mathematical induction on $M \geqslant 1$:

$$\begin{aligned} \bigoplus^M Bf &= B(\mathrm{id} + (\mathrm{id} - B) + (\mathrm{id} - B)^2 + \cdots + (\mathrm{id} - B)^{M-1})\, f \\ &= \mathbf{b}^T (I + (I - \mathscr{B}) + (I - \mathscr{B})^2 + \cdots + (I - \mathscr{B})^{M-1})\, \mathbf{f}. \end{aligned} \tag{21}$$

Here we used the notation $\mathbf{f} = (\lambda_i f)_{i=0}^{n}$, $\mathbf{b} = (b_i)_{i=0}^{n}$ as above and id for the identity on X. Equation (21) shows that iterated Boolean sums have a close relation to partial sums of the formal Neumann series for $\mathscr{B}^{-1} = (I-(I-\mathscr{B}))^{-1}$. Now (21) and Proposition 1 with $A := I-\mathscr{B}$ make up the main part of the proof.

What is left to prove is (6) and (7). We know that for $\varrho(I-\mathscr{B})<1$ the Gramian $\mathscr{B}$ is nonsingular, and mathematical induction on $M \geqslant 1$ shows

$$\bigoplus^M Bf = \mathbf{b}^T \mathscr{B}^{-1}(I-(I-\mathscr{B})^M)\mathbf{f}. \tag{22}$$

Now (22) yields (7) for $M \to \infty$.

In order to prove (6) we proceed as follows: Let $\{l_0, ..., l_n\} \in \operatorname{span}\{b_0, ..., b_n\}$ be the biorthonormal basis to the linear functionals $\{\lambda_0, ..., \lambda_n\}$, i.e. we have

$$\lambda_i l_j = \delta_{ij}, \qquad i, j = 0, ..., n, \tag{23}$$

whereas δ_{ij} is the Kronecker symbol. (Such a basis exists since $\mathscr{B}$ is nonsingular.) The last equation implies the relation

$$\mathbf{b}^T \mathscr{B}^{-1} = \boldsymbol{l}^T, \tag{24}$$

where $\boldsymbol{l}$ is the vector of Lagrange functions $\boldsymbol{l} = (l_0, ..., l_n)^T$. To see this, just apply the functionals λ_i, $i = 0, ..., n$, to (24), write this in matrix form and use (23) to obtain $\mathscr{B}\mathscr{B}^{-1} = I$. Now (24) yields

$$L^{\oplus} f = \lim_{M \to \infty} \bigoplus^M Bf = \mathbf{b}^T \mathscr{B}^{-1} \mathbf{f} = \boldsymbol{l}^T \mathbf{f}$$

and finally

$$\lambda_i (f - L^{\oplus} f) = \lambda_i f - \lambda_i (\mathbf{l}^T \mathbf{f}) = 0, \qquad i = 0, ..., n,$$

by the biorthonormality of the functionals λ_i and the Lagrange functions l_i. The uniqueness of the interpolant is a direct consequence of the nonsingularity of $\mathscr{B}$. ∎

Proof of Theorem 3. The idea here is the following: First, prove that $\mathscr{B}$ is semiconvergent and that $\varrho(I-\mathscr{B})<1$ holds. With this, apply Theorems 1 and 2.

1. *$\mathscr{B}$'s eigenvalues are* $0 < l_{-k+1} < \cdots < l_{n-2} = l_{n-1} = 1$, *hence* $\varrho(I-\mathscr{B})<1$!

Observe that $\mathscr{B}$ has the following structure:

$$\mathscr{B} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & * \\ \vdots & & & & \vdots \\ * & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Here, all the "$*$"-entries are nonnegative. Hence, the eigenvalues of $\mathcal{B}$ are 1 (twofold) and eigenvalues that are at the same time eigenvalues of the matrix $\tilde{\mathcal{B}} := (N_{j,k,t}(\tau_i))_{i,j=-k+2}^{n-2}$. From de Boor [2] (Corollary 2), we can see that $\tilde{\mathcal{B}}$ is totally nonnegative, i.e. all of $\tilde{\mathcal{B}}$'s minors of any order are nonnegative. Moreover, $\tilde{\mathcal{B}}$ is nonsingular since $\mathcal{B}$ is nonsingular and has the special structure we mentioned above.

By our assumptions on the interpolation nodes and the spline knots in (12), we know that for $|i-j| \leqslant 1$ we have $N_{j,k,t}(\tau_i) > 0$, $i, j = -k+2, ..., n-2$. Thus we can apply Proposition 6 to $\tilde{\mathcal{B}}$, which shows that $\tilde{\mathcal{B}}$ has $n+k-3$ distinct positive eigenvalues. Since $\tilde{\mathcal{B}}$ is also nonnegative and irreducible by (12), we can apply Proposition 7 to see that $\varrho(\tilde{\mathcal{B}}) < 1$. (Remember that $N_{-k+1,k,t}(\tau_{-k+2}) > 0$, hence the $(-k+2)$th row sum of $\tilde{\mathcal{B}}$ is less than 1.) This proves that $\mathcal{B}$'s eigenvalues are $0 < l_{-k+1} < \cdots < l_{n-2} = l_{n-1} = 1$.

2. *$\mathcal{B}$ is semiconvergent*!

In order to show this, we use the characterization for semiconvergent matrices from Proposition 4: From the discussion in 1. above, we see that $\varrho(\mathcal{B}) = 1$ and that 1 is the only eigenvalue of modulus 1. From the discussion in 3. below we see that we have two linearly independent eigenvectors with respect tot eigenvalue 1, hence $\operatorname{rank}(I-\mathcal{B})^2 = \operatorname{rank}(I-\mathcal{B})$.

3. *Finding the part of* $\mathbf{f}$ *that belongs to eigenvector* 1 *for* $\tau_i = t_i^*$, $i = -k+1, ..., n-1$.

Remember that Schoenberg's variation diminishing spline approximant to $f \in C([t_0, t_n])$ is given by

$$Vf := \sum_{i=-k+1}^{n-1} f(t_i^*) N_{i,k,t}$$

and reproduces linear functions. This implies that $(1)_{i=-k+1}^{n-1}$ and $(t_i^*)_{i=-k+1}^{n-1}$ are eigenvectors of $\mathcal{B}$ with respect to eigenvalue 1. From this, we see that every eigenvector with respect to 1 must have the form $(f_{-k+1} + (t_i^* - t_{-k+1}^*)/(t_{n-1}^* - t_{-k+1}^*)(f_{n-1} - f_{-k+1}))_{i=-k+1}^{n-1}$.

Now if we split $\mathbf{f} = (f(t_i^*))_{i=-k+1}^{n-1}$ according to $\mathbf{f} = \mathbf{f}_1 + (\mathbf{f} - \mathbf{f}_1)$ with $\mathbf{f}_1 = (f(t_{-k+1}^*) + (t_i^* - t_{-k+1}^*)/(t_{n-1}^* - t_{-k+1}^*)(f(t_{n-1}^*) - f(t_{-k+1}^*)))_{i=-k+1}^{n-1}$, we have $(\mathbf{f} - \mathbf{f}_1)_{-k+1} = (\mathbf{f} - \mathbf{f}_1)_{n-1} = 0$. Thus $\mathbf{f}_1$ is an eigenvector with respect to 1, and the components of $\mathbf{f} - \mathbf{f}_1$ for the eigenvectors corresponding to eigenvalue 1 are zero.

4. *Applying Theorems* 1 *and* 2.

Because $\mathcal{B}$ is semiconvergent, we can apply Theorem 1 to see that $\lim_{M \to \infty} B_I^M f$ exists for all $f \in C[t_0, t_n]$.

In order to prove (i) of the theorem here, we remark that the Schoenberg nodes t_i^* satisfy (12). Now the discussion in 3. above and Theorem 1 show

$$\lim_{m \to \infty} B_I^M f = \sum_{i=-k+1}^{n-1} \left(f(t_{-k+1}^*) + \frac{t_i^* - t_{-k+1}^*}{t_{n-1}^* - t_{-k+1}^*} \left(f(t_{n-1}^*) - f(t_{-k+1}^*) \right) \right) N_{i,k,t}$$

$$= f(t_{-k+1}) + \frac{\cdot - t_{-k+1}^*}{t_{n-1}^* - t_{-k+1}^*} \left(f(t_{n-1}^*) - f(t_{-k+1}^*) \right).$$

The second equation above is a consequence of the linear precision of Schoenberg's variation diminishing spline approximant.

Now it remains to prove (ii). From 1. above, we know that $\varrho(I - \mathscr{B}) < 1$ holds. Theorem 2 tells us that then the Boolean sums $\bigoplus^M B_I f$ converge and that we have

$$f(\tau_i) = L_I^{\oplus} f(\tau_i), \qquad i = -k+1, ..., n-1,$$

by (6). This concludes the proof. ∎

Proof of Theorem 4. Sablonnière has shown in [16], Theorem 3, that B_{II} is a self-adjoint operator in $L_2[t_0, t_n]$ with real, positive and simple eigenvalues

$$0 < l_{-k+1} < \cdots < l_{n-1} = 1. \tag{25}$$

Since B_{II} is linear and has finite dimensional range, these eigenvalues are also the eigenvalues of the corresponding Gramian $\mathscr{B} = (k/(t_{i+k} - t_i) \int_{t_0}^{t_n} N_{i,k,t}(x) N_{j,k,t}(x)\, dx)_{i,j=-k+1}^{n-1}$. Therefore, $\mathscr{B}$ is nonsingular and—by Proposition 3—semiconvergent, and it satisfies $\varrho(I - \mathscr{B}) = 1 - l_{-k+1} < 1$. Thus Theorems 1 and 2 show the existence of $\lim_{M \to \infty} B_{II}^M f$ and $\lim_{m \to \infty} \bigoplus^M B_{II} f$.

In order to see (i), note that $\mathbf{e} := (1, ..., 1)^T$ is an eigenvector to eigenvalue 1 and that the selfadjointness of B_{II} implies that there is an orthonormal basis of eigenvectors of $\mathscr{B}$ in $\mathbb{R}^{n+k-1}$. The part of $\mathbf{f} = (k/(t_{i+k} - t_i) \int_{t_0}^{t_n} N_{i,k,t}(x) f(x)\, dx)_{i=-k+1}^{n-1}$ that corresponds to eigenvalue 1 is therefore

$$(\mathbf{e}^T \mathbf{f} / \mathbf{e}^T \mathbf{e}) \mathbf{e} = 1/(n+k-1) \sum_{i=-k+1}^{n-1} k/(t_{i+k} - t_i) \int_{t_0}^{t_n} N_{i,k,t}(x) f(x)\, dx \cdot \mathbf{e}.$$

Together with the partition of unity property for B-splines, this yields (i).

To see (ii), we apply (6) to the present case and obtain

$$k/(t_{i+k} - t_i) \int_{t_0}^{t_n} N_{i,k,t}(x)(f(x) - g(x))\, dx = 0, \qquad i = -k+1, ..., n-1,$$

with $g := \lim_{m \to \infty} \bigoplus^M B_{II} f$. Hence the difference $f - g$ is orthogonal to $\text{span}\{N_{i,k,t} : i = -k+1, ..., n-1\}$ with respect to the scalar product in $L_2[t_0, t_n]$, and the theorem is proved. ▮

Proof of Corollary 5. What is left to proof is (iii). This follows from (5), (8) and the spectral properties of Durrmeyer operators which were studied by Derriennic (see [7], Prop. I.2 and Th. III.3). ▮

Proof of Theorem 6. The tensor product Schoenberg operator B_T of (14) can be written as

$$B_T = B_1 \hat{\otimes} B_2$$

with

$$B_1 f := \sum_{i=-k+1}^{n-1} f(\tau_i) N_{i,k,t}, \qquad f \in C([t_0, t_n]),$$

$$B_2 g := \sum_{j=-\tilde{k}+1}^{\tilde{n}-1} g(\tilde{\tau}_j) N_{j,\tilde{k},\tilde{t}}, \qquad g \in C([\tilde{t}_0, \tilde{t}_{\tilde{n}}]),$$

and where $\hat{\oplus}$ denotes the completion of the tensor product with respect to the so-called ε-crossnorm (see [12] for further details). Convergence of tensor product operators was investigated by Haußmann and Zeller in [12]. We adopt Theorem 1 thereof in the following form.

PROPOSITION 8 (Haußmann, Zeller). *Assume that we are given continuous operators K, K_M: $C([t_0, t_n]) \to C([t_0, t_n])$ and L, L_M: $C([\tilde{t}_0, \tilde{t}_{\tilde{n}}]) \to C([\tilde{t}_0, \tilde{t}_{\tilde{n}}])$ as well as a function $f \in C([t_0, t_n] \times [\tilde{t}_0, \tilde{t}_{\tilde{n}}])$. Then with the ε-completion of the tensor product, $\hat{\otimes}$, we have*

$$\|(K_M \hat{\otimes} L_M) f - (K \hat{\otimes} L) f\|_\infty \leqslant \|K\|_\infty \cdot \max_{x \in [t_0, t_n]} \|L f_x - L_M f_x\|_\infty$$
$$+ \|L_M\|_\infty \cdot \max_{y \in [\tilde{t}_0, \tilde{t}_{\tilde{n}}]} \|K f_y - K_M f_y\|_\infty, \quad (26)$$

where $f_x(y) := f(x, y)$, x fixed, and $f_y(x) := f(x, y)$, y fixed.

Remark. The norms $\|\cdot\|_\infty$ in (26) are supremum norms on $C([t_0, t_n])$, $C([\tilde{t}_0, \tilde{t}_{\tilde{n}}])$ and $C([t_0, t_n] \times [\tilde{t}_0, \tilde{t}_{\tilde{n}}])$ as well as operator norms. It should be clear which one is meant in either case.

In order to prove (i) of Theorem 6, we set $K := \lim_{M \to \infty} B_1^M$ and $L := \lim_{M \to \infty} B_2^M$ as well as $K_M := B_1^M$ and $L_M := B_2^M$ in (26) with B_1 and B_2 as defined above. Since K is the operator that maps a continuous function onto its two point Lagrange interpolant with respect to t_0 and t_n, we know $\|K\|_\infty = 1$. Moreover, we have

$$\|L_M\|_\infty = \|B_2^M\|_\infty \leqslant \|B_2\|_\infty^M = 1,$$

as is well known.

Now from the univariate result (Theorem 3) we know that $\|Lf_x - L_M f_x\|_\infty \to 0$ for every fixed x and $\|Kf_y - K_M f_y\|_\infty \to 0$ for every fixed y. In what follows we will show that this convergence is uniform. To this end, we write $\mathbf{b} = (N_{i,k,t})_{i=-k+1}^{n-1}$ as well as $\mathscr{B} = (N_{j,k,t}(t_i^*))_{i,j=-k+1}^{n-1}$ and $\mathbf{f}_y := (f(t_i^*, y))_{i=-k+1}^{n-1}$, where $t_i^* = (t_{i+1} + \cdots + t_{i+k-1})/(k-1)$ are the Schoenberg knots again. Then we have the unique representation $\mathbf{f}_y = \mathbf{f}_1 + \mathbf{f}_0$ with $\mathbf{f}_1$ being the eigenvector part with respect to eigenvalue 1 and $\mathscr{B}$. From Theorems 1 and 3 and the respective proofs we conclude

$$\begin{aligned}\|Kf_y - K_M f_y\|_\infty &= \|\mathbf{b}^T\mathbf{f}_1 - \mathbf{b}^T\mathscr{B}^{M-1}(\mathbf{f}_1 + \mathbf{f}_0)\|_\infty \\ &= \|\mathbf{b}^T\mathbf{f}_1 - \mathbf{b}^T\mathbf{f}_1 - \mathbf{b}^T\mathscr{B}^{M-1}\mathbf{f}_0\|_\infty \\ &= \|\mathbf{b}^T\mathscr{B}^{M-1}\mathbf{f}_0\|_\infty.\end{aligned}$$

The stability of the B-spline basis (see deBoor [3], Cor XI.1) now implies

$$\|Kf_y - K_M f_y\|_\infty \leqslant \|\mathscr{B}^{M-1}\mathbf{f}_0\|_\infty, \tag{27}$$

where the norm on the right hand side is the vector maximum norm, defined by $\|\mathbf{a}\|_\infty := \max_i |a_i|$ for a vector $\mathbf{a}$. From Proposition 4 and the proof of Theorem 3 as well as the definition of the vector $\mathbf{f}_0$ above, we can see

$$\mathscr{B}^{M-1}\mathbf{f}_0 = V\begin{pmatrix}0 & 0\\ 0 & \mathscr{K}^{M-1}\end{pmatrix}V^{-1}\mathbf{f}_0,$$

where $\mathscr{K} = \mathrm{diag}(l_{n-3}, ..., l_{-k+1})$ is the diagonal matrix whose nonzero entries are the eigenvalues of $\mathscr{B}$ that are different from 1 with $l_{n-3} > \cdots > l_{-k+1}$. This, together with (27) yields

$$\begin{aligned}\|Kf_y - K_M f_y\|_\infty &\leqslant \|V\|_\infty \left\|\begin{pmatrix}0 & 0\\ 0 & \mathscr{K}^{M-1}\end{pmatrix}\right\|_\infty \|V^{-1}\|_\infty \|\mathbf{f}_0\|_\infty \\ &= \|V\|_\infty \cdot l_{n-3}^{M-1} \cdot \|V^{-1}\|_\infty \|\mathbf{f}_0\|_\infty,\end{aligned} \tag{28}$$

where for a square matrix $A = (a_{ij})$, the matrix maximum norm is defined to be $\|A\|_\infty := \max_i \sum_j |a_{ij}|$.

From the proof of Theorem 3 we get

$$\|\mathbf{f}_0\|_\infty = \left\| \left(f(t_i^*, y) - \left(\frac{t_{n-1}^* - t_i^*}{t_{n-1}^* - t_{-k+1}^*} f(t_{-k+1}^*, y) \right.\right.\right.$$
$$\left.\left.\left. + \frac{t_i^* - t_{-k+1}^*}{t_{n-1}^* - t_{-k+1}^*} f(t_{n-1}^*, y) \right) \right)_{i=-k+1}^{n-1} \right\|_\infty$$
$$\leqslant 3 \max_{-k+1 \leqslant i \leqslant n-1} |f(t_i^*, y)|$$
$$\leqslant 3 \|f\|_\infty .$$

This, together with (28), shows

$$\|Kf_y - K_M f_y\|_\infty \leqslant C l_{n-3}^{M-1} \|f\|_\infty$$

whereas the constant C is independent of y. Thus, we have

$$\max_{y \in [\tilde{t}_0, \tilde{t}_{\tilde{n}}]} \|Kf_y - K_M f_y\|_\infty \to 0, \tag{29}$$

and analogously

$$\max_{x \in [t_0, t_n]} \|Lf_x - L_M f_x\|_\infty \to 0. \tag{30}$$

Now (26) together with some basic tensor product theory, for which a convenient reference is the book of Greub [11], yields the relation

$$(K \hat{\otimes} L) f = \lim_{M \to \infty} (B_1^M \hat{\otimes} B_2^M) f = \lim_{M \to \infty} (B_1 \hat{\otimes} B_2)^M f = \lim_{M \to \infty} B_T^M f,$$

which equals (i).

In order to prove (ii) we set $K := \lim_{M \to \infty} \bigoplus^M B_1$, $L := \lim_{M \to \infty} \bigoplus^M B_2$, $K_M := \bigoplus^M B_1$, and $L_M := \bigoplus^M B_2$. Then K is a continuous linear operator by our conditions for the knots and nodes, as can be seen from the proof of Theorem 3. Thus, we have $\|K\|_\infty < \infty$. Moreover,

$$\|L_M\|_\infty = \|\mathrm{id} - (\mathrm{id} - B_2)^M\|_\infty$$
$$\leqslant 1 + \|\mathrm{id} - B_2\|_\infty^M \leqslant 2,$$

since we have seen $\|\mathrm{id} - B_2\|_\infty < 1$ in Theorem 3. Here, id is the identity in $C([\tilde{t}_0, \tilde{t}_{\tilde{n}}])$.

As above, we have $\|Lf_x - L_M f_x\|_\infty \to 0$ for every fixed x and $\|Kf_y - K_M f_y\|_\infty \to 0$ for every fixed y. In order to prove the uniform convergence of $\|Kf_y - K_M f_y\|_\infty \to 0$ for $y \in [\tilde{t}_0, \tilde{t}_{\tilde{n}}]$, we remember

$$\begin{aligned}\|Kf_y - K_M f_y\|_\infty &= \|\mathbf{b}^T \mathscr{B}^{-1} \mathbf{f}_y - \mathbf{b}^T \mathscr{B}^{-1}(I - (I - \mathscr{B})^M) \mathbf{f}_y\|_\infty \\ &= \|\mathbf{b}^T \mathscr{B}^{-1}(I - \mathscr{B})^M \mathbf{f}_y\|_\infty\end{aligned}$$

from the proof of Theorem 2, where the notations for $\mathbf{b}$, $\mathscr{B}$, and $\mathbf{f}_y$ are analogous to above. With the techniques from the first part of the present proof we can conclude

$$\begin{aligned}\|Kf_y - K_M f_y\|_\infty &\leqslant \|\mathscr{B}^{-1}(I - \mathscr{B})^M \mathbf{f}_y\|_\infty \\ &\leqslant \|\mathscr{B}^{-1}\|_\infty \|(I - \mathscr{B})^M\|_\infty \|\mathbf{f}_y\|_\infty \\ &\leqslant \|\mathscr{B}^{-1}\|_\infty \|(I - \mathscr{B})^M\|_\infty \|f\|_\infty\end{aligned}$$

Again, the right hand side is independent of $y \in [\tilde{t}_0, \tilde{t}_{\tilde{n}}]$. Since $\varrho(I - \mathscr{B}) < 1$ according to the proof of Theorem 3, this implies $\|(I - \mathscr{B})^M\|_\infty \to 0$ and shows the uniform convergence for iterated Boolean sums of Schoenberg type operators. Thus, by means of (26), we have shown (ii). ∎

Proof of Theorem 7. The main part of the proof here is to determine the eigenvalues of the Gramian $\mathscr{B} = (b_{\tilde{\imath}\tilde{\jmath}}(i/(k-1),\ j/(k-1)))_{0 \leqslant i+j,\, \tilde{\imath}+\tilde{\jmath} < k}$, by considering the Gramian $\mathscr{G} = ((i/(k-1))^{\tilde{\imath}}\ (j/(k-1))^{\tilde{\jmath}})_{0 \leqslant i+j,\, \tilde{\imath}+\tilde{\jmath} < k}$ which is similar to $\mathscr{B}$. In order to determine the eigenvalues of $\mathscr{G}$, we mimic the technique of Kelisky and Rivlin in [13]: Observe that after some straightforward calculations the bivariate Bernstein operator $B_{I,\Delta}$ can be written in the following form in terms of monomials:

$$\begin{aligned}B_{I,\Delta}(f, x, y) &= \sum_{0 \leqslant i+j < k} f\left(\frac{i}{k-1}, \frac{j}{k-1}\right) \\ &\quad \times \frac{(k-1)!}{i!\, j!\, (k-1-i-j)!}\, x^i y^j (1-x-y)^{k-1-i-j} \\ &= \sum_{q=0}^{k-1} \sum_{\mu=0}^{q} \left(\sum_{i=0}^{\mu} \sum_{j=0}^{q-\mu} f\left(\frac{i}{k-1}, \frac{j}{k-1}\right) \right. \\ &\quad \left. \times \frac{(k-1)!}{i!\, j!\, (\mu-i)!\, (q-\mu-j)!\, (k-1-q)!} (-1)^{q-i-j} \right) x^\mu y^{q-\mu}.\end{aligned} \tag{31}$$

After some further calculations, for $0 \leqslant \tilde{\imath}+\tilde{\jmath}<k$ we see

$$B_{I,\Delta}(x^{\tilde{\imath}}y^{\tilde{\jmath}}, x, y) = \sum_{q=0}^{k-1} \sum_{\mu=0}^{q} \left(\sum_{i=0}^{\mu} \sum_{j=0}^{q-\mu} \left(\frac{i}{k-1}\right)^{\tilde{\imath}} \left(\frac{j}{k-1}\right)^{\tilde{\jmath}} \cdot \frac{(k-1)!}{i!\,j!\,(\mu-i)!\,(q-\mu-j)!\,(k-1-q)!} (-1)^{q-i-j} \right) x^{\mu} y^{q-\mu}$$

$$= \sum_{q=0}^{k-1} \sum_{\mu=0}^{q} \left(\frac{(k-1)!}{(k-1-q)!} \frac{1}{(k-1)^{\tilde{\imath}+\tilde{\jmath}}} \left\{ {\tilde{\imath} \atop \mu} \right\} \left\{ {\tilde{\jmath} \atop q-\mu} \right\} \right) x^{\mu} y^{q-\mu}, \tag{32}$$

where the symbol $\left\{ {u \atop v} \right\}$ stands for the Stirling numbers of the second kind (see Graham et al. [10], Ch. 6.1, for the necessary background). Since we have (see [10], p. 264) $\left\{ {u \atop v} \right\}=0$ for $v>u$ and $\left\{ {u \atop v} \right\}=1$, equation (32) gives

$$B_{I,\Delta}(x^{\tilde{\imath}}y^{\tilde{\jmath}}, x, y) = \sum_{\mu=0}^{\tilde{\imath}} \sum_{v=0}^{\tilde{\jmath}} \frac{(k-1)!}{(k-1-\mu-v)!} \frac{1}{(k-1)^{\tilde{\imath}+\tilde{\jmath}}} \left\{ {\tilde{\imath} \atop \mu} \right\} \left\{ {\tilde{\jmath} \atop v} \right\} x^{\mu} y^{v}. \tag{33}$$

Thus, the matrix $\mathscr{G}=(g_{(i,j),(\tilde{\imath},\tilde{\jmath})})_{0 \leqslant i+j, \tilde{\imath}+\tilde{\jmath}<k}$ that describes the operation of the triangular Bernstein operator on the monomial basis of π_k^2 is an upper triangular matrix. Its eigenvalues equal its diagonal entries which can be read off as

$$l_{(\tilde{\imath},\tilde{\jmath})} = \frac{(k-1)!}{(k-1-\tilde{\imath}-\tilde{\jmath})!} \frac{1}{(k-1)^{\tilde{\imath}+\tilde{\jmath}}} \leqslant 1, \qquad 0 \leqslant \tilde{\imath}+\tilde{\jmath}<k,$$

with the biggest eigenvalues $l_{(0,0)}=l_{(1,0)}=l_{(0,1)}=1$ and the smallest eigenvalues $l_{(k-1,0)}=\cdots=l_{(0,k-1)}=(k-1)!/(k-1)^{k-1}$. Since $\mathscr{G}$ is similar to $\mathscr{B}$, we know that these are also the eigenvalues of $\mathscr{B}$.

The rest of the proof can be obtained by using the fact that the bivariate Bernstein operator has the linear precision property and by using the same techniques as in the proof of Theorem 3. ∎

Proof of Theorem 8. From Derriennic [8] (Prop. I.1 and Th. II.1) we know that $B_{II,\Delta}$ is a self-adjoint operator in $L_2(S)$ with eigenvalues $l_{(0,0)}=1$ and $l_{(i,j)}=(k+1)!\,(k-1)!/((k+1+i+j)!\,(k-1-i-j)!)$, $0<i+j<k$. With this, the proof goes along the same lines as the proofs of Theorem 4 and Corollary 5. ∎

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